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Continuous Additive Functionals of a Markov Process with Applications to Processes with Independent Increments*

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INTRODUCTION AND PRELIMINARIES

The first part of the present paper is devoted to the study of continuous additive functionals (CAF's) of a Markov process satisfying Hunt's hypothesis (A). In the second part we apply these general results to study the "restriction" of a Markov process to a finite set. This is accomplished by making use of the "local time" for X in the given finite set. In particular, one can compute explicitly the probability that a stable process on the real line with index $\alpha > 1$ hits a before b starting from x. See Corollary 6.8 for the explicit formula. The reader primarily interested in the applications should turn directly to section six. Most of the material in section one is contained in the author's University of Hamburg Lecture Notes [1] under slightly more restrictive hypotheses. The material in Section 3 was obtained in collaboration with R. M. Blumenthal and the author wishes to thank Professor Blumenthal for allowing it to appear here.

The reader is referred to the author's expository paper [2] for all definitions, notations, and terminology. However for the convenience of the reader we repeat some of the basic definitions. Let E be a locally compact separable metric space and let $E_\Delta = E \cup \{\Delta\}$ where Δ is a point adjoined to E as the point at ∞ if E is not compact and as an isoltaed point if E is compact. Let $X = (\Omega, X_t, \theta_t, P^x)$ be a Hunt process with state space $(E, \mathscr{B})$. That is $X_t : \Omega \to E_\Delta$ for each t, $0 \leqslant t \leqslant \infty$, such that $X_\infty(\omega) = \Delta$, $t \to X_t(\omega)$ is right continuous and has left hand limits on $[0, \infty)$, and $X_s(\omega) = \Delta$ for all $s \geqslant t$ if $X_t(\omega) = \Delta$. The *shift operators* θ_t are maps from Ω to Ω such that $X_t \circ \theta_h = X_{t+h}$. Let $\mathscr{F}^0(\mathscr{F}_t^0)$ be the smallest σ-algebra in Ω for which the maps $\{X_s; s \leqslant \infty\}$ $(\{X_s; s \leqslant t\})$ are measurable. For each x in E_Δ, P^x is a probability measure on $\mathscr{F}^0$ satisfying $x \to P^x(\Lambda)$, is $\mathscr{B}_\Delta$ measurable for each Λ in $\mathscr{F}^0$, and $P^x(X_0 = x) = 1$. Here $\mathscr{B}_\Delta(\mathscr{B})$ is the σ-algebra of

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Borel sets in $E_\Delta(E)$. Let $\mathscr{F}$ denote the intersection of the P^μ completion of $\mathscr{F}^0$ taken over all finite measures μ on $\mathscr{B}_\Delta$. Let $\mathscr{F}_t$ be the σ-algebra of all sets Λ such that for each μ there exists Λ_μ in $\mathscr{F}_t^0$ with $\Lambda \, \Delta \, \Lambda_\mu$ in $\mathscr{F}$ and $P^\mu(\Lambda \Delta \Lambda_\mu) = 0$, where "$\Delta$" denotes symmetric difference. It is assumed that X is strong Markov (stopping times are relative to $\mathscr{F}_{t+}$ unless explicitly stated otherwise), and that X is quasi-left continuous, that is, whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T then $X(T_n) \to X(T)$ almost surely on $\{T < \zeta\}$. Here $\zeta = \inf\{t : X_t = \Delta\}$ is the lifetime of X and almost surely means almost surely with respect to each P^x. Under these conditions $\mathscr{F}_{t+} = \mathscr{F}_t$.

A continuous additive function, CAF, of X is a family $A = \{A(t); t \geqslant 0\}$ of nonnegative random variables on $(\Omega, \mathscr{F})$ such that:

(i) $A(0) = 0$, $t \to A(t)$ is continuous and nondecreasing, and $A(s) = \lim_{t \uparrow \zeta} A(t)$ whenever $s \geqslant \zeta$; each of these statements holding a.s.

(ii) $A(t)$ is $\mathscr{F}_t$ measurable for each t.

(iii) For each t, s one has almost surely

$$A(t + s, \omega) = A(t, \omega) + A(s, \theta_t \omega).$$

Meyer proved in [3] that A has the strong Markov property, that is (iii) above continues to hold when t is replaced by any stopping time T and s by any nonnegative random variable S. We refer the reader to [2] for additional properties of additive functionals and notation.

1. The Fine Support of a CAF

Let A be a *continuous additive functional* of X. We will assume that $A(t)$ is almost surely finite on $\{\zeta > t\}$. Define

$$R = \inf\{t : A(t) > 0\} = \sup\{t : A(t) = 0\}. \tag{1.1}$$

It is immediate that R is a stopping time and the continuity of A implies that $A(R) = 0$ almost surely. In particular $R = \infty$ a.s. if and only if $t \to A(t, \omega)$ is the zero function almost surely, that is $A = 0$.

1.2. Lemma. *Let T be a stopping time, then $T + R \circ \theta_T = R$ almost surely on $\{T < R\}$, and if $\varphi(x) = E^x(e^{-R})$, then φ is 1-excessive.*

Proof. Using the strong Markov property for A one has almost surely

$$\begin{aligned} R[\theta_T \omega] &= \inf\{s : A(s, \theta_T \omega) > 0\} \\ &= \inf\{s : A(s + T) - A(T) > 0\} \\ &= \inf\{s \geqslant T(\omega) : A(s) - A(T) > 0\} - T(\omega). \end{aligned}$$

Therefore $T+R\circ\theta_T\geqslant R$ with equality on $\{T<R\}$, both statements holding almost surely. It also follows easily that $t+R\circ\theta_t\to R$ as $t\to 0$ almost surely. These facts imply without difficulty that φ is 1-excessive.

We next define

$$F=\{x:\varphi(x)=1\}=\{x:P^x(R=0)=1\}. \tag{1.3}$$

In view of Lemma 1.2, F is nearly Borel measurable and finely closed. We will call F the *fine support* of A. By the zero-one law $P^x(R>0)=1$ if x is not in F.

1.3. THEOREM. *$R=T_F$ almost surely.*

PROOF. Let $T=T_F$. By Lemma 1.2, $R\circ\theta_T=R-T$ on $\{T<R\}$. Hence for each x

$$\begin{aligned} P^x[T<R] &= P^x[T<R;\, R\circ\theta_T>0;\, T<\infty] \\ &\leqslant E^x\{P^{X(T)}[R>0];\, T<\infty\} \\ &= 0, \end{aligned}$$

since F being finely closed implies that $X(T)$ is in F almost surely on $\{T<\infty\}$.

Let F^r be the set of points regular for F. We next show that $P^x[R<T]=0$ provided x is not in $F\setminus F^r$. This is obvious if x is in F^r, and so we need only consider the case x not in F. Since $A(R)=0$ we have for any $t>0$

$$\begin{aligned} P^x[R<T] &= P^x[A(R+t)>0;\, R<T] \\ &= P^x[A(t,\theta_R)>0;\, R<T] \\ &= E^x\{P^{X(R)}[A(t)>0];\, R<T\}. \end{aligned}$$

But x is not in F, and so $P^x(R>0)=1$. Consequently $X(R)$ is not in F, P^x almost surely on $\{R<T\}$. Moreover if y is not in F

$$P^y[A(t)>0]\leqslant P^y[R<t]\to 0$$

as $t\to 0$. Hence $P^x[R<T]=0$ provided x is not in $F\setminus F^r$.

It follows from what we have proved so far that $P^x[R=T]=1$ provided x is not in $F\setminus F^r$. If $\varphi(x)=E^x(e^{-R})$ and $\psi(x)=E^x(e^{-T})$, then φ and ψ are both 1-excessive and agree except possibly on $F\setminus F^r$. But $F\setminus F^r$ is semipolar, and hence φ and ψ must agree everywhere. Consequently $F=F^r$ (F^r is contained in F since F is finely closed), and this completes the proof of Theorem 1.3.

1.4. COROLLARY. *$A=0$ if and only if F is empty.*

We now introduce the functional inverse to A,

$$\tau(t,\omega)=\inf\{s:A(s,\omega)>t\}=\sup\{s:A(s,\omega)=t\}. \tag{1.5}$$

It is immediate that τ is right continuous and *strictly increasing* almost surely on $\{t : \tau(t) < \infty\}$. The following facts are well known and easy to check, see [1] or [3]:

(i) Each $\tau(t)$ is a stopping time and

$$\tau(t + s, \omega) = \tau(t, \omega) + \tau(s, \theta_{\tau(t)}\omega), \qquad \text{a.s.} \tag{1.6}$$

(ii) If f is a bounded Borel function from $[0, \infty]$ to $[0, \infty]$ with $f(\infty) = 0$, then

$$\int_0^\infty f(t)\, dA(t) = \int_0^\infty f[\tau(t)]\, dt \qquad \text{a.s.}$$

In particular note that $\tau(0) = R$.

Define, the following sets, each of which depends on ω:

$$\begin{aligned} I &= \{t : A(t + \epsilon) - A(t) > 0 \text{ for all } \epsilon > 0\}, \\ \bar{I} &= \{t : A(t + \epsilon) - A(t - \epsilon) > 0 \text{ for all } \epsilon > 0\}, \\ Z &= \{t : X(t) \in F\}, \\ Q &= \{t < \infty; \tau(s) = t \text{ for some } s\}. \end{aligned}$$

In the definition of $\bar{I}$, $A(u)$ is understood to be zero if $u < 0$. Clearly $\bar{I}$ is the closure of I almost surely and $I = Q$ almost surely.

1.7. THEOREM. $Q = I \subset Z \subset \bar{I}$ *almost surely.*

PROOF. In view of the above remark we need only show $Q \subset Z \subset \bar{I}$.

(a) $Z \subset \bar{I}$. If $T = T_F$ it is clear that

$$\{\omega : Z \not\subset \bar{I}\} \subset \bigcup_{r<s} \{A(s) - A(r) = 0; r + T \circ \theta_r < s\}$$

where the union is over all pairs (r, s) of rationals with $0 \leqslant r < s$. But for any x

$$P^x[A(s) - A(r) = 0; r + T \circ \theta_r < s] = E^x\{P^{X(r)}[A(s - r) = 0; T < s - r]\},$$

and Theorem 1.3 implies that this last expression is zero.

(b) $Q \subset Z$. Recall that $F = \{\varphi = 1\}$ where $\varphi(x) = E^x(e^{-R})$ and that $R = \tau(0)$. We now compute for a fixed t and x

$$\begin{aligned} E^x\varphi[X_{\tau(t)}] &= E^x\{E^{X[\tau(t)]}(e^{-R}); \tau(t) < \infty\} \\ &= E^x\{\exp[-\tau(0, \theta_{\tau(t)})]; \tau(t) < \infty\} \\ &= E^x\{\exp[-\tau(0 + t) + \tau(t)]; \tau(t) < \infty\} \\ &= P^x[\tau(t) < \infty]. \end{aligned}$$

But $\varphi \leqslant 1$ and hence $\varphi[X_{\tau(t)}] = 1$ almost surely on $\{\tau(t) < \infty\}$ for each fixed t. However φ is 1-excessive and so $t \to \varphi[X_{\tau(t)}]$ is right continuous almost surely. Combining these facts we obtain that for each x

$$P^x\{\varphi[X_{\tau(t)}] < 1 \text{ for some } t \text{ with } \tau(t) < \infty\} = 0$$

or

$$P^x\{X_{\tau(t)} \notin F \text{ for some } t \text{ with } \tau(t) < \infty\} = 0.$$

That is, $Q \subset Z$ almost surely, and so the proof of Theorem 1.7 is complete.

If $f \geqslant 0$ is nearly Borel measurable we will write fA for the family of random variables $\int_0^t f(X_s)\, dA(s)$. It is immediate that fA is a continuous additive functional provided $(fA)(t)$ is finite almost surely on $\{\zeta > t\}$. For example under our assumptions on A this will certainly be the case if f is bounded.

1.8. Corollary. *Let I_F be the indicator function of F, then $A = I_F A$.*

Proof. As usual the equality of additive functionals means equivalence. Consider an ω such that $t \to A_t(\omega)$ is continuous and increasing and $I(\omega) \subset Z(\omega) \subset \bar{I}(\omega)$. For such an ω the measure $dA_t(\omega)$ on $[0, \infty)$ is supported by $\bar{I}(\omega)$. Moreover $\bar{I}(\omega) \setminus I(\omega)$ is countable and hence has $dA_t(\omega)$ measure zero. Thus for a fixed t

$$A(t, \omega) = \int_{[0,t]\cap I(\omega)} dA_s(\omega) = \int_{[0,t]\cap I(\omega)} I_F[X_s(\omega)]\, dA_s(\omega)$$

$$= \int_0^t I_F[X_s(\omega)]\, dA_s(\omega).$$

Since the complement of the set of ω's in question has P^x measure zero for all x, this proves that A and $I_F A$ are equivalent.

We say that A *vanishes* on a nearly Borel set D provided $I_D A = 0$, and that A vanishes on an *arbitrary* set D provided that A vanishes on all nearly Borel subsets of D. Note that if A has a finite λ-potential for some $\lambda \geqslant 0$, then A vanishes on a nearly Borel set D if and only if $U_A^\lambda I_D = 0$. Since E has a countable base for its topology it is immediate that there exists a smallest closed set, which we will call the support of A, on whose complement A vanishes.

1.9. Corollary. *F is the smallest finely closed set on whose complement A vanishes.*

Proof. Corollary 1.8 implies that A vanishes on $E_\Delta \setminus F$. (Note that any A vanishes on $\{\Delta\}$.) Suppose there exists a finely open set D on which A vanishes and such that $D \cap F$ is not empty. If y is in $D \cap F$, then there exists a nearly Borel set B contained in D and containing y such that $P^y(T_{B^c} > 0) = 1$. But A vanishes on B and is continuous, and so

$$A(T_{B^c}) = \int_0^{T_{B^c}} I_{B^c}(X_s)\, dA(s) = 0$$

almost surely. Hence $T_{B^c} \leqslant R$ almost surely and this leads to a contradiction since y in F implies that $P^y(R = 0) = 1$.

In view of Corollary 1.9 we will call F the *fine support* of A.

1.10. Corollary. (i) *The closure of F is the support of A.*

(ii) *A is strictly increasing if and only if $F = E$.*

Proof. (i) If A vanishes on an open set G and $G \cap \bar{F}$ is not empty, then $G \cap F$ is not empty and one obtains the same contradiction as in the proof of Corollary 1.9.

(ii) To say that A is strictly increasing means, of course, that $t \to A(t)$ is strictly increasing on $[0, \zeta]$ almost surely. If A is strictly increasing then obviously $P^x(R = 0) = 1$ for all x in E, and hence $F = E$. Conversely if $E = F$, then for any x and $s > t$

$$\begin{aligned} P^x[A(s) - A(t) = 0; s \leqslant \zeta] &\leqslant P^x[A(s-t, \theta_t) = 0; t < \zeta] \\ &= E^x\{P^{X(t)}[A(s-t) = 0]; t < \zeta\} \end{aligned}$$

and this last expression equals zero since X_t is in $E = F$ when $t < \zeta$. It is now immediate that $t \to A_t(\omega)$ is strictly increasing on $[0, \zeta(\omega)]$ almost surely.

Remark. If X satisfies Hunt's hypotheses (F) and (G) and μ is a measure on E charging no semipolar set with a finite potential, it then follows on combining the results of this section with those of Meyer [3] that μ has a fine support. In particular, if X satisfies (F), (G), and (H) (for example Brownian motion in three or more dimensions), then any measure with a finite potential has a fine support. This might be of interest in classical potential theory.

2. The Potential Operators of a CAF

In this section we will assume that A is a CAF and that A has a finite λ-potential for some fixed $\lambda \geqslant 0$. That is

$$u_A^\lambda(x) = E^x \int_0^\infty e^{-\lambda t}\, dA(t)$$

is finite for all x. We define the λ-potential operator, $U_A{}^\lambda$, associated with A by

$$U_A{}^\lambda f(x) = E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dA(t).$$

This is well defined if f is nearly Borel measurable and either bounded or nonnegative. In particular, $u_A{}^\lambda = U_A{}^\lambda 1$ is called the λ-potential of A. Our notation differs slightly from that used in [2] or [3], namely, we use $U_A{}^\lambda$ for the *potential operator* and $u_A{}^\lambda$ for the *potential* of A. The operator $f \to U_A{}^\lambda f$ is given by a kernel which we denote by $U_A{}^\lambda(x, dy)$, that is,

$$U_A{}^\lambda f(x) = \int U_A{}^\lambda(x, dy) f(y).$$

It is an immediate consequence of Corollary 1.8 that for each x in E the measure $U_A{}^\lambda(x, \cdot)$ is concentrated on F. Therefore $U_A{}^\lambda f$ depends only on the restriction of f to F. Also it follows from Corollary 1.9 that if Γ is a nearly Borel set contained in F such that $U_A{}^\lambda(x, \Gamma) = 0$ for all x, then $F \setminus \Gamma$ is finely dense in F.

It will be necessary to consider $U_A{}^\lambda f$ when f is only assumed to be universally measurable and bounded (or nonnegative). It is immediate that

$$U_A{}^\lambda f(x) = \int U_A{}^\lambda(x, dy) f(y)$$

is well defined and universally measurable in x for such f. Moreover standard considerations show that $t \to f[X_t(\omega)]$ is measurable with respect to the ν_ω completion of $\mathscr{T}$ almost surely where $\mathscr{T}$ is the σ-algebra of Borel sets of $[0, \infty)$ and ν_ω is the measure on $\mathscr{T}$ defined by $\nu_\omega(\Gamma) = \int_\Gamma dA(t, \omega)$. It is now straightforward that

$$U_A{}^\lambda f(x) = E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dA_t$$

for such f and that $U_A{}^\lambda f$ is λ-excessive if $f \geqslant 0$. Let $\mathbf{M}$ be the bounded universally measurable functions on E, then it follows from the above discussion that $U_A{}^\lambda f$ is finely continuous and nearly Borel measurable for each f in $\mathbf{M}$. We let $\mathscr{A}$ denote the σ-algebra of universally measurable sets.

2.1. Proposition. *If Γ is in $\mathscr{A}$, then $U_A{}^\lambda(x, \Gamma) = 0$ for all x in E if and only if $U_A{}^\lambda(x, \Gamma) = 0$ for all x in F.*

PROOF. Suppose $U_A{}^\lambda(\cdot, \Gamma)$ vanishes on F, then for any x in E we have using Theorem 1.3

$$\begin{aligned} U_A{}^\lambda(x, \Gamma) &= E^x \int_0^\infty e^{-\lambda t} I_\Gamma(X_t)\, dA(t) \\ &= E^x \left\{ \int_{T_F}^\infty e^{-\lambda t} I_\Gamma(X_t)\, dA(t);\ T_F < \infty \right\} \\ &= E^x\{e^{-\lambda T_F}\, U_A{}^\lambda(X_{T_F}, \Gamma);\ T_F < \infty\}, \end{aligned}$$

and this last expression is zero since $X(T_F)$ is in F almost surely on $\{T_F < \infty\}$.

Let $\mathbf{M}(F)$ be the bounded $\mathscr{A}$ measurable functions on F. (F is nearly Borel measurable and hence is in $\mathscr{A}$.) Let $\mathbf{C}(F)$ be the bounded *nearly Borel measurable* functions on F that are finely continuous on F, that is, those bounded nearly Borel measurable functions on F that are continuous when F is given the relative topology it inherits as a subspace of E equipped with the fine topology. Naturally $\mathbf{M}(F)$ and $\mathbf{C}(F)$ are Banach spaces under the usual supremum norm, and $\mathbf{C}(F)$ is a closed subspace of $\mathbf{M}(F)$. It will be convenient to regard any f in $\mathbf{M}(F)$ as being defined on all of E_Δ and having the value zero on $E_\Delta \setminus F$. The operator $U_A{}^\lambda$ induces an operator W^λ on $\mathbf{M}(F)$ in the obvious manner

$$W^\lambda f(x) = I_F(x)\, U_A^\lambda f(x)$$

The following proposition is an immediate consequence of the preceding discussion.

2.2. PROPOSITION. *If* $u_A{}^\lambda(x) = U_A{}^\lambda 1(x)$ *is bounded, then* $W^\lambda \mathbf{M}(F)$ *is contained in* $\mathbf{C}(F)$.

2.3. LEMMA. *If f is in $\mathbf{C}(F)$ and $\tau(t)$ is the functional inverse to $A(t)$, then $f[X_{\tau(t)}] \to f(x)$ as $t \to 0$ almost surely P^x for each x in F.*

PROOF. According to Theorem 1.7 we may assume that $X[\tau(t)]$ is in F whenever $\tau(t) < \infty$ by throwing out a set of ω's that is almost surely null (i.e., has P^x measure zero for all x). Also $\tau(t) \downarrow \tau(0) = 0$ almost surely P^x if x is in F (since $\tau(0) = R$). Let x be an element of F, then for a given $\epsilon > 0$ there exists a nearly Borel measurable fine neighborhood, N, of x such that $|f(y) - f(x)| < \epsilon$ for all y in $N \cap F$. By definition $P^x[T_{N^c} > 0] = 1$. Thus for P^x almost all ω we have $\tau(t, \omega) < T_{N^c}(\omega)$ for all sufficiently small $t \geqslant 0$. But this implies that $X[\tau_t(\omega), \omega]$ is in N for all sufficiently small strictly positive t, and since $X[\tau(t)]$ is in F whenever $\tau(t) < \infty$ it follows that $|f(X_{\tau(t)}) - f(x)| < \epsilon$ for all sufficiently small t almost surely P^x. Hence Lemma 2.3 is established.

If f is in $\mathbf{M}$ we define

$$Q_t^\lambda f(x) = E^x\{e^{-\lambda\tau(t)} f[X_{\tau(t)}]\}.$$

It follows from Theorem 1.7 that Q_t^λ depends only on the restriction of f to F and it is clear that Q_t^λ maps $\mathbf{M}$ into $\mathbf{M}$ if u_A^λ is bounded. Let T_t^λ be the operator on $\mathbf{M}(F)$ induced by Q_t^λ, that is $T_t^\lambda f = I_F Q_t^\lambda f$ for any f in $\mathbf{M}(F)$.

2.4. THEOREM. *Let u_A^λ be bounded, then:*

(a) *$\{Q_t^\lambda; t \geqslant 0\}$ and $\{T_t^\lambda; t \geqslant 0\}$ are semigroups of bounded operators on $\mathbf{M}$ and $\mathbf{M}(F)$ respectively.*

(b) *$T_t^\lambda(x) \to f(x)$ as $t \to 0$ for all f in $\mathbf{C}(F)$.*

(c) *$U_A^\lambda f = \int_0^\infty Q_t^\lambda f dt$ and $W^\lambda f = \int_0^\infty T_t^\lambda f dt$.*

(d) *If $\{J^\mu\}$ is hte resolvent of $\{T_t^\lambda\}$, then $J^\mu : \mathbf{C}(F) \to \mathbf{C}(F)$ for any $\mu \geqslant 0$, and $J^0 = W^\lambda$.*

(e) *W^λ is one-to-one on $\mathbf{C}(F)$.*

PROOF. Statement (a) is a straigthforward consequence of property (1.6) of $\tau(t)$. Statement (b) follows easily from Lemma 2.3. Also

$$\begin{aligned} U_A^\lambda f(x) &= E^x \int_0^\infty e^{-\lambda t} f[X_t]\, dA(t) \\ &= E^x \int_0^\infty e^{-\lambda\tau(t)} f[X_{\tau(t)}]\, dt \\ &= \int_0^\infty Q_t^\lambda f(x)\, dt \end{aligned}$$

and so (c) is established. As to (d), since J^0 exists we have from the resolvent equation that $J^\mu - J^0 = -\mu J^0 J^\mu$. If f is in $\mathbf{C}(F)$ then $J^\mu f$ is in $\mathbf{M}(F)$, and so, using (c), $J^\mu f = W^\lambda[f - \mu J^\mu f]$ which is in $\mathbf{C}(F)$ according to Proposition 2.2.

Finally it remains to prove (e). Suppose f is in $\mathbf{C}(F)$ and $W^\lambda f = 0$, then the resolvent equation implies that $J^\mu f = 0$ for all $\mu \geqslant 0$. Hence

$$0 = \mu J^\mu f(x) = \mu \int_0^\infty e^{-\mu t}\, T_t^\lambda f(x)\, dt.$$

But $T_t^\lambda f(x) \to f(x)$ as $t \to 0$ and so $\mu J^\mu f(x) \to f(x)$ as $\mu \to \infty$. Therefore $f = 0$ and Theorem 2.4 is established.

2.5. REMARK. There is an obvious analog of Theorem 2.4 if one merely assumes that u_A^λ is finite rather than bounded. We omit a detailed statement.

2.6. REMARK. Since W^λ is one-to-one on $\mathbf{C}(F)$, $(W^\lambda)^{-1}$ exists as a linear transformation with domain $\mathbf{D}$ in $\mathbf{C}(F)$. Moreover $\mathbf{D} = W^\lambda\mathbf{C}(F) = J^\mu\mathbf{C}(F)$ for any $\mu \geqslant 0$ (the last equality for $\mu > 0$ is a consequence of the resolvent equation). Therefore if f is in $\mathbf{C}(F)$, $\mu J^\mu f$ is in $\mathbf{D}$ and $\mu J^\mu f \to f$ pointwise as $\mu \to \infty$. Thus $\mathbf{D}$ is dense in $\mathbf{C}(F)$ in the topology of pointwise convergence. Finally, if $D = (-W^\lambda)^{-1} = -(J^0)^{-1}$, it follows from the resolvent equation that $(\mu - D)^{-1} = J^\mu$ for each $\mu \geqslant 0$. The operator D is the weak infinitesimal generator of the process $X[\tau(t)]$ "killed at time $A(S^\lambda)$", where S^λ is an exponentially distributed random variable with parameter λ (i.e., $P^x[S^\lambda > t] = e^{-\lambda t}$ for all $t \geqslant 0$ and all x) that is completely independent of the process X.

3. CAF's with Given Fine Support

In Section 1 it was shown that the fine support F of a CAF was a finely closed nearly Borel subset of E with the property that each x in F is regular for F. It is natural to ask if given such a set F does there exist a CAF whose fine support is F. We will not answer this question in complete generality. However, we will give an affirmative answer under certain additional conditions on X and F. These additional conditions are satisfied in many applications. In the remainder of this section we will assume that the following condition holds:

$$P_t\mathbf{C}_0(E) \subset \mathbf{C}_0(E) \qquad \text{for each} \qquad t \geqslant 0, \tag{C}$$

where $\mathbf{C}_0(E)$ denotes the Banach space of continuous functions on E vanishing at infinity. Since the paths are right continuous it follows from (C) that $\{P_t\}$ is strongly continuous on $\mathbf{C}_0(E)$. Therefore, according to Blumenthal's theorem [4], if $\{T_n\}$ is an increasing sequence of stopping times with limit T, then $X(T_n) \to X(T)$ almost surely on $\{T < \infty\}$.

3.1. THEOREM. *Let (C) hold. Let F be a finely closed nearly Borel subset of E with compact closure, $\bar{F}$, in E with the property that each x in F is regular for F. If, in addition, $\bar{F} \setminus F$ is polar, then there exists a continuous additive functional A whose fine support is F. Moreover A can be chosen so that $U_A^\lambda 1$ is bounded for each $\lambda > 0$.*

PROOF. Let $T = T_F$ and set $\varphi(x) = E^x(e^{-T})$. Clearly φ is a bounded 1-excessive function. In order to show the existence of a continuous additive functional A such that $\varphi = U_A^1 1$ we will make use of a theorem of Šur [5].[1]

[1] Šur considers only the case $\lambda = 0$. However the general case may be reduced to this case by standard considerations. See, for example, [1].

Let $B_n = \{x : \varphi(x) - P^1_{1/n}\varphi(x) \geqslant \epsilon\}$ where $P_t^\lambda = e^{-\lambda t} P_t$, and let $T_n = T_{B_n}$; then Šur's theorem states that φ is the 1-potential of a CAF if for each $\epsilon > 0$, $P^x[T_n < S] \to 0$ as $n \to \infty$ for each x, where S is an exponentially distributed random variable with parameter 1 that is independent of the process X; that is, $P^x[S > t] = e^{-t}$ for all x and $t \geqslant 0$. We now proceed to the verification of Šur's condition.

First of all suppose G is an open set containing $\bar{F}$. Condition (C) implies that the hypotheses of Lemma 2.1 of [4] hold. Hence for any $\epsilon > 0$ there exists $t_0 > 0$ such that $P^x[T > t_0] \geqslant 1 - \epsilon$ for all x in $E_\Delta \setminus G$. But

$$\varphi(x) - P_t^1\varphi(x) \leqslant P^x[T \leqslant t \wedge S] \leqslant P^x[T \leqslant t] < \epsilon$$

if $t \leqslant t_0$ and x is in $E_\Delta \setminus G$. Therefore $\varphi - P_t^1\varphi$ approaches zero as $t \to 0$ uniformly on $E_\Delta \setminus G$. Consequently for a given $\epsilon > 0$, $B_n \subset G$ for n sufficiently large. Now let $\{G_j\}$ be a decreasing sequence of neighborhoods of $\bar{F}$ such that $\cap\, \bar{G}_j = \bar{F}$. Since $B_{n+1} \subset B_n$ for all n, $\{T_n\}$ is an increasing sequence of stopping times. Let $Q = \lim T_n$. But $X(T_n)$ is in B_n almost surely on $\{T_n < \infty\}$ and $B_n \subset \bar{G}_j$ for each j and sufficiently large n. Also $X(T_n) \to X(Q)$ almost surely on $\{Q < \infty\}$ and hence $X(Q)$ is in $\bar{G}_j$ for all j. That is, $X(Q)$ is in $\bar{F}$ almost surely on $\{Q < \infty\}$. The facts that each B_n is finely closed and $\cap\, B_n$ is empty imply that $Q > 0$ almost surely, and consequently $X(Q)$ is in F almost surely on $\{Q < \infty\}$, $\bar{F} \setminus F$ being polar by assumption.

The fact that $e^{-t}\,\varphi(X_t)$ is a right continuous supermartingale with respect to each P^x implies that

$$\lim_n E^x\{e^{-T_n}\,\varphi(X_{T_n})\} \geqslant E^x\{e^{-Q}\,\varphi(X_Q)\}.$$

On the other hand

$$\begin{aligned} E^x\{e^{-T_n}\,\varphi(X_{T_n})\} &= E^x\{\varphi(X_{T_n});\ T_n < S \\ &= E^x\{\varphi(X_{T_n});\ T_n < S < Q] + E^x\{\varphi(X_{T_n});\ T_n < S; Q \leqslant S\}. \end{aligned}$$

Since φ is bounded and $T_n \to Q$, the first term above approaches zero as $n \to \infty$, while the second term is dominated by $P^x(Q \leqslant S) = E^x(e^{-Q})$. But $X(Q)$ is in F almost surely on $\{Q < \infty\}$, and $\varphi(y) = 1$ if y is in F. Therefore $E^x(e^{-Q}) = E^x\{e^{-Q}\varphi(X_Q)\}$. Combining this with the previous inequality yields

$$\lim_n E^x\{e^{-T_n}\,\varphi(X_{T_n})\} = E^x\{e^{-Q}\,\varphi(X_Q)\},$$

and it is well known ([6, Prop. 5.2] or [1, Lemma 2.3.2]) that this last equality implies Šur's condition. Thus, invoking Šur's Theorem there exists a continuous additive functional A such that

$$\varphi(x) = E^x \int_0^\infty e^{-t}\, dA(t).$$

It remains to show that the fine support of A is F. Recall that $\varphi(x) = E^x(e^{-T})$ where $T = T_F$, and as usual let $R = \sup\{t : A(t) = 0\}$. We first note that

$$P_F^1\varphi(x) = E^x\{e^{-T}\varphi(X_T)\} = E^x(e^{-T}) = \varphi(x),$$

since X_T is in F almost surely on $\{T < \infty\}$ and each point in F is regular for F. Also

$$P_F^1\varphi(x) = E^x \int_T^\infty e^{-t}\, dA(t),$$

and consequently

$$E^x \int_0^T e^{-t}\, dA(t) = 0$$

for all x. But this implies that $T \leqslant R$ almost surely. On the other hand $R = T + R \circ \theta_T$ on $\{T < R\}$, and so

$$P^x[T < R] = P^x[R \circ \theta_T > 0;\ T < R] \leqslant E^x[P^{X(T)}(R > 0);\ T < \infty].$$

For each y in F we have

$$1 = \varphi(y) = E^y \int_0^\infty e^{-t}\, dA(t) = E^y \int_R^\infty e^{-t}\, dA(t)$$
$$= E^y\{e^{-R}\, \varphi(X_R)\} \leqslant E^y(e^{-R}),$$

since $\varphi \leqslant 1$. Therefore $P^y(R = 0) = 1$ if y is in F, and hence $P^x(T < R) = 0$ for all x. Thus $R = T$ almost surely. By definition the fine support of A is $\{x : P^x(R = 0) = 1\}$ and consequently is the same as $\{x : P^x(T = 0) = 1\} = F$. This completes the proof of Theorem 3.1.

3.2. Remark. Using the notations of the previous theorem one can show easily that $U_A{}^\lambda 1 = \varphi - (\lambda - 1)\, U^\lambda\varphi$. See, for example, Theorem 2.1.2 of [1].

3.3. Remark. An equivalent formulation of Theorem 3.1. is the following: Let (C) hold. If K is a compact subset of E such that $K \setminus K^r$ is polar, then there exists a continuous additive functional A whose fine support is K^r.

3.4. Remark. If F consists of a single point, or more generally is a finite set such that each x in F is regular for F, then without assuming that (C) holds one can show by a slight variant of the above argument that there exists a CAF, A, whose fine support is F and such that

$$E^x(e^{-T_F}) = E^x \int_0^\infty e^{-t}\, dA(t).$$

See [7].

4. CAF's with Finite Fine Support

The following theorem is analogous to a result of Motoo [8] (see also [1, Theorem 2,4.6]).

4.1. Theorem. *Let A be a CAF and suppose that its fine support, F, is countable. Let $\lambda \geqslant 0$ be fixed and assume that $u_A^\lambda = U_A^\lambda 1$ is finite, then a necessary and sufficient condition that a λ-excessive function f have the representation $f = U_A^\lambda h$ with h a Borel measurable function on E satisfying $0 \leqslant h \leqslant 1$ is that $f \leqslant U_A^\lambda 1$ and that*

$$f(x) - P_t^\lambda f(x) \leqslant E^x \int_0^t e^{-\lambda u}\, dA(u)$$

for all t and x.

Proof. Since the necessity of this condition is clear we proceed to the proof of its sufficiency. Using the condition one sees easily that $g = u_A^\lambda - f$ is λ-excessive. Since u_A^λ is a regular λ-potential (see [2] for the terminology) and $f + g = u_A^\lambda$, it follows that both f and g are regular λ-potentials. Hence there exist continuous additive functionals B and C such that $f = U_B^\lambda 1$ and $g = U_C^\lambda 1$ and $A = B + C$. Thus if T is a stopping time

$$0 \leqslant f(x) - P_T^\lambda f(x) = E^x \int_0^T e^{-\lambda u}\, dB(u) \leqslant E^x \int_0^T e^{-\lambda u}\, dA(u).$$

Let $\tau(t)$ be the inverse of A, then $A[\tau(t)] = t$ on $\{\tau(t) < \infty\}$ and $A(\infty) \leqslant t$ if $\tau(t) = \infty$, so that $A[\tau(t)] \leqslant t$ in all cases. Hence

$$0 \leqslant f(x) - P_{\tau(t)}^\lambda f(x) \leqslant E^x \int_0^{\tau(t)} e^{-\lambda u}\, dA(u) \leqslant t. \tag{4.2}$$

Define J^μ as in Section 2, that is,

$$\begin{aligned} J^\mu f(x) &= E^x \int_0^\infty e^{-\mu t}\, e^{-\lambda\tau(t)}\, g(X_{\tau(t)})\, dt \\ &= \int_0^\infty e^{-\mu t}\, P_{\tau(t)}^\lambda g(x)\, dt. \end{aligned}$$

In particular $\{J^\mu; \mu \geqslant 0\}$ satisfies the resolvent equation and $J^0 = U_A^\lambda$. Taking Laplace transforms in (4.2) we obtain

$$0 \leqslant \mu[f - \mu J^\mu f] \leqslant 1$$

for each $\mu > 0$. This clearly implies that $\mu J^\mu f \to f$ as $\mu \to \infty$. Let $g_\mu = \mu[f - \mu J^\mu f]$, then using the diagonal procedure one can find a sequence

$\{\mu_n\}$ with $\mu_n \to \infty$ such that $g_{\mu_n}(x)$ converges for each x in the countable set F. Define $h(x)$ to be this limit if x is in F and $h(x)$ to be zero if x is not in F. Clearly $0 \leqslant h \leqslant 1$ and h is Borel measurable.

Using the resolvent equation we obtain

$$U^{\lambda}{}_{A} g_{\mu_n} = \mu_n[J^0 - \mu_n J^0 J^{\mu_n} f] = \mu_n J^{\mu_n} f \to f$$

as $n \to \infty$. On the other hand $U_A{}^{\lambda}(x, \cdot)$ is concentrated on F for each x, and since $\{g_{\mu_n}\}$ converges boundedly to h on F we find that $U_A{}^{\lambda} g_{\mu_n} \to U_A{}^{\lambda} h$. Combining these facts yields Theorem 4.1.

4.3. COROLLARY. *Let A satisfy the hypotheses of Theorem 4.1. If there exist additive functionals B and C such that $A = B + C$, then $B = hA$ with $0 \leqslant h \leqslant 1$, h Borel measurable.*

PROOF. First of all it follows that B and C must be continuous. Also

$$u_B{}^{\lambda}(x) - P_t{}^{\lambda} u_B{}^{\lambda}(x) = E^x \int_0^t e^{-\lambda u}\, dB(u) \leqslant E^x \int_0^t e^{-\lambda u}\, dA(u),$$

and so Theorem 4.1 yields $u_B{}^{\lambda} = U_A{}^{\lambda} h$. Now the uniqueness theorem for CAF's implies that $B = hA$.

Let x_0 be a point of E such that x_0 is regular for $\{x_0\}$, then, as remarked at the end of Section 3, there exists a continuous additive functional A_{x_0} whose fine support is $\{x_0\}$ and such that

$$E^x \int_0^{\infty} e^{-t}\, dA_{x_0}(t) = E^x(e^{-T})$$

where $T = T_{\{x_0\}}$. See [7] for a proof. The CAF, A_{x_0}, is called the local time at x_0. We have the following uniqueness result.

4.4. THEOREM. *Let x_0 be as above. If A is a CAF with fine support $\{x_0\}$ and with finite λ-potential for some $\lambda \geqslant 0$, then $A = bA_{x_0}$ where b is a positive constant.*[2]

PROOF. Let $A_0 = A_{x_0}$ and define $B = A + A_0$, then B has fine support $\{x_0\}$ and finite λ-potential for some $\lambda > 0$ (A_0 has bounded λ-potential for any $\lambda > 0$). Let $U_A{}^{\lambda}(x, \cdot) = c(x)\, \epsilon_0$ and $U^{\lambda}_{A_0}(x, \cdot) = d(x)\, \epsilon_0$, so that

$$U_B{}^{\lambda}(x, \cdot) = [c(x) + d(x)]\, \epsilon_0$$

[2] One can show by a more careful analysis that if A is a CAF with fine support $\{x_0\}$ and such that $A(t)$ is almost surely finite for each finite t, then A must have a bounded λ-potential for any positive λ.

where ϵ_0 is unit mass at x_0. Also Corollary 4.3 implies that $A = fB$ and $A_0 = gB$ with $0 < f,\ g \leqslant 1$ and $f(x_0) + g(x_0) = 1$. Consequently $U_A^\lambda(x, dy) = U_B^\lambda(x, dy) f(y)$ and hence $c(x) = [c(x) + d(x)] f(x_0)$. But $0 < f(x_0) < 1$ since $A \neq 0$ and $A_0 \neq 0$, and so $c(x) = bd(x)$ where $b = f(x_0)[1 - f(x_0)]^{-1} > 0$. Therefore $U_A^\lambda(x, \cdot) = bU_{A_0}^\lambda(x, \cdot)$ and so an application of the uniqueness theorem for CAF's completes the proof of Theorem 4.4.

Let $F = \{x_1, \cdots, x_n\}$ be a finite subset of E such that each point of F is regular for F, which is clearly equivalent to requiring that x_i is regular for $\{x_i\}$, $1 \leqslant i \leqslant n$. Let A_i denote the local time at $\{x_i\}$. Suppose that A is a CAF whose fine support is F and having finite λ-potential for some $\lambda \geqslant 0$, then if f_i is the characteristic function of $\{x_i\}$ one has $A = \sum_{i=1}^n f_i A$. But $f_i A = b_i A_i$ for suitable positive constants b_i according to Theorem 4.4. Thus $A = \sum_{i=1}^n b_i A_i$ for appropriate positive constants b_i. Of course, each A_i is only determined up to a multiplicative constant. Thus there is "essentially" only one CAF with F as its fine support.

5. The Imbedded Process

Let $F = \{x_1, \cdots, x_n\}$ and suppose that x_i is regular for $\{x_i\}$ for each i. Let $T_i = T_{\{x_i\}}$ and A_i be the local time at $\{x_i\}$. Let $u_i^\lambda = U_{A_i}^\lambda 1$ be the λ-potential of A_i, $\lambda > 0$. Let $A = \sum A_i$ so that $u_A^\lambda = \sum u_i^\lambda$. We will now apply the results of Section 2 to A. First of all for each $\lambda > 0$ define a *matrix* U_A^λ by

$$U_A^\lambda(i,j) = U_A^\lambda(x_i, \{x_j\}) = \sum_k U_{A_k}^\lambda(x_i, \{x_j\}) = U_{A_j}^\lambda(x_i, \{x_j\}) = u_j^\lambda(x_i). \tag{5.1}$$

We will use the symbol U_A^λ both for the matrix introduced above and the operator on $\mathbf{M}(F)$ introduced in Section 2 which is just the operator induced by the matrix U_A^λ on the *finite* dimensional space $\mathbf{M}(F)$. In the present case $\mathbf{M}(F) = \mathbf{C}(F)$ and both are finite dimensional. According to Theorem 2.4, U_A^λ is one-to-one on $\mathbf{C}(F) = \mathbf{M}(F)$ and hence in the present case the matrix U_A^λ is invertible. We define

$$Q^\lambda = -(U_A^\lambda)^{-1}, \qquad \lambda > 0, \tag{5.2}$$

As in Section 2 we introduce the semigroups ($\lambda \geqslant 0$)

$$Q_t^\lambda f(x_i) = E^{x_i}\{e^{-\lambda\tau(t)} f[X_{\tau(t)}]\}$$

where $\tau(t)$ is the inverse of A. Since $X_{\tau(t)}$ is in F almost surely on $\{\tau(t) < \infty\}$, $\{Q_t^\lambda;\ t \geqslant 0\}$ is a semigroup of operators on $\mathbf{M}(F)$ for each $\lambda \geqslant 0$, and we may, and will, regard $\{Q_t^\lambda;\, t \geqslant 0\}$ as a semigroup of matrices. It follows from Theorem 2.4 that $Q_t^\lambda \to I$ as $t \to 0$ for each $\lambda > 0$. Explicitly

$$Q_t^\lambda(i,j) = E^{x_i}\{e^{-\lambda\tau(t)};\, X[\tau(t)] = x_j\},$$

so that the matrix Q_t^λ is Laplace transform of the joint distribution of $\tau(t)$ and $X[\tau(t)]$.

The process $X[\tau(t)]$ has the following intuitive description. Since $\tau(t)$ is in F almost surely on $\{\tau(t) < \infty\}$, the process $X[\tau(t)]$ is "just" the process $X(t)$ "sampled" when it is in F. It is immediate that $X[\tau(t)]$ is a Markov chain with state space $F \cup \{\Delta\}$ and we will call it the imbedded process (or chain). The joint process $\{X[\tau(t)], \tau(t)\}$ is an example of what Neveu [9] has called a process of type F.

Returning now to the matrices Q_t^λ, we have that $U_A^\lambda = \int_0^\infty Q_t^\lambda dt$ if $\lambda > 0$ and $Q_t^\lambda \to I$ as $t \to 0$. Moreover $Q^\lambda = -(U_A^\lambda)^{-1}$ exists. It follows easily from these relations that $Q_t^\lambda = e^{tQ^\lambda}$. See, for example, [10, Sections 9.4 and 11.2]. Also one can give a simple minded proof using exactly the same argument as in the proof of Theorem 2.1 of [7]. Moreover

$$Q_t(i,j) = P^{x_i}\{X[\tau(t)] = x_j\}$$

is a semigroup of sub-Markovian matrices such that $Q_t \to I$ as $t \to 0$. Therefore there exists a matrix Q such that $Q_t = e^{tQ}$. Indeed $Q = \lim_{t\downarrow 0} t^{-1}[Q_t - I]$, and thus if $Q = (q_{ij})$ then $q_{ij} \geqslant 0$ if $i \neq j$ and $q_{ii} \leqslant 0$.

5.3. Proposition. *Using the notation developed above*

$$\lim_{\lambda\to 0} Q^\lambda = Q.$$

Proof. It is immediate from the definition that $0 \leqslant Q_t^\lambda \leqslant Q_t$ and that $Q_t^\lambda \uparrow Q_t$ as $\lambda \downarrow 0$. (Inequalities among matrices are elementwise.) If $\lambda_0 > 0$ is fixed, then $I - Q_t \leqslant I - Q_t^\lambda \leqslant I - Q_t^{\lambda_0}$, $0 < \lambda \leqslant \lambda_0$. Choose $t_0 > 0$ such that $\| I - Q_{t_0}^{\lambda_0} \| \leqslant (2n)^{-1}$ and $\| I - Q_{t_0} \| \leqslant (2n)^{-1}$ where n is the cardinality of F, and if $B = (b_{ij})$ is a matrix $\| B \| = \sup_i \sum_j | b_{ij} |$. These inequalities clearly yield $\| I - Q_{t_0}^\lambda \| \leqslant \frac{1}{2}$ if $\lambda \leqslant \lambda_0$. Now it is easy to see (and also well known) that

$$Q^\lambda = -t_0^{-1} \sum_{n=1}^{\infty} n^{-1}(I - Q_{t_0}^\lambda)^n, \qquad \lambda \leqslant \lambda_0,$$

with a similar expression for Q. Therefore

$$Q - Q^\lambda = -t_0^{-1} \sum_{n=1}^{\infty} n^{-1}[(I - Q_{t_0})^n - (I - Q_{t_0}^\lambda)^n]$$

if $\lambda \leqslant \lambda_0$. It is now immediate using the bounded convergence theorem that $Q^\lambda \to Q$ as $\lambda \to 0$.

One can now apply Neveu's theory [9] of F processes to the process $\{X[\tau(t)], \tau(t)\}$ to obtain various interesting formulas. However we leave this task to the interested reader. We close this section with the following theorem on which the applications in Section 6 are based.

5.4. THEOREM. *Using the notation developed above let T_i' be the first hitting time of $F \setminus \{x_i\}$ by X, then*

$$P^{x_i}[X(T_i') = x_j] = -\lim_{\lambda\to 0} \frac{q^\lambda(i,j)}{q^\lambda(i,i)} = -\frac{q(i,j)}{q(i,i)},$$

$i \neq j$. Of course, the $q^\lambda(i,j)$ are the elements of Q^λ and $q(i,j)$ those of Q.

PROOF. Fix i and let $R = T_i'$. Let $Y_t = X[\tau(t)]$ be the imbedded Markov chain with state space $F \cup \{\Delta\}$. Define $H = \inf\{t > 0 : Y(t) \neq x_i\}$. We will first show that $H = R$, P^{x_i} almost surely on $\{Y(H) \in F\}$. Since $Y(t)$ has right continuous paths H is strictly positive and hence $\tau(H) > 0$ almost surely P^{x_i}. Also $X[\tau(H)] = Y(H)$ is in $F \setminus \{x_i\}$ almost surely on $\{Y(H) \in F\}$. Therefore $R \leqslant \tau(H) < \infty$ almost surely P^{x_i} on $\{Y(H) \in F\}$. Consider an ω such that $0 < R(\omega) < \tau_{H(\omega)}(\omega)$. Since X_R is in $F \setminus \{x_i\}$ on $\{R < \infty\}$, it must be the case that $R(\omega)$ is not in the range of $u \to \tau_u(\omega)$ and consequently $R(\omega)$ is not a point of right increase of $t \to A(t, \omega)$. Thus

$$\{0 < R < \tau(H)\} \subset \bigcup_n \left\{A\left(R + \frac{1}{n}\right) = A(R); R < \infty\right\}.$$

Now

$$P^{x_i}\left[A\left(R + \frac{1}{n}\right) - A(R) = 0; R < \infty\right]$$

$$= E^{x_i}\left\{P^{X(R)}\left[A\left(\frac{1}{n}\right) = 0\right]; R < \infty\right\} = 0,$$

since $X(R)$ is in $F \setminus \{x_i\}$ on $\{R < \infty\}$. Also $P^{x_i}(R = 0) = 0$ and so it follows that $R = \tau(H) < \infty$ almost surely P^{x_i} on $\{Y(H) \in F\}$. Moreover the above argument actually shows that R is a point of right increase of A almost surely P^{x_i} on $\{R < \infty\}$. Hence P^{x_i} almost surely on $\{R < \infty\}$ one has $\{Y(H) \in F\}$, and so $\{R < \infty\} = \{Y(H) \in F\}$ almost surely P^{x_i}.

It is a standard fact in the theory of Markov chains that

$$P^{x_i}[Y(H) = x_j] = -\frac{q(i,j)}{q(i,i)}.$$

Therefore, in view of what was proved above, we obtain

$$P^{x_i}[X(R) = x_j] = -\frac{q(i,j)}{q(i,i)}$$

and recalling the definition of R and using Proposition 5.3 yields Theorem 5.4.

6. Applications

In this section X will be a real valued Hunt process with stationary independent increments. Therefore as in well known

$$E^x(e^{iyX(t)}) = e^{iyx}\, e^{-t\psi(y)}$$

where

$$\psi(y) = imy + \frac{\sigma^2}{2} y^2 + \int_{-\infty}^{\infty} \left[1 - e^{iyu} - \frac{iyu}{1+u^2}\right] \nu(du) \tag{6.1}$$

with m a real number, $\sigma^2 \geqslant 0$, and ν a nonnegative measure satisfying $\int_{-\infty}^{\infty} x^2(1 + x^2)^{-1}\, \nu(dx) < \infty$. We assume that $m = 0$ and $\sigma^2 = 0$. In addition if ψ_R is the real part of ψ we assume that

$$\int_{-\infty}^{\infty} [\lambda + \psi_R(x)]^{-1}\, dx < \infty \tag{6.2}$$

for all positive λ. Under assumption (6.2) each x is regular for $\{x\}$ (see [7]). Let

$$p(t, x, y) = f(t, y - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iz(y-x)}\, e^{-t\psi(z)}\, dz$$

be the transition density of X with respect to Lebesgue measure and

$$U^\lambda(y - x) = \int_0^\infty e^{-\lambda t}\, p(t, x, y)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iz(y-x)}}{\lambda + \psi(z)}\, dz$$

be the potential kernel for X. All these integrals exist absolutely under assumption (6.2). It was shown in [7, Sec. 3] that under the present assumptions one can choose the local time A_{x_0} at x_0 so that

$$U^\lambda(x_0 - x) = E^x \int_0^\infty e^{-\lambda t}\, dA_{x_0}(t) \tag{6.3}$$

holds for all x and $\lambda > 0$.

Let $F = \{x_1, \cdots, x_n\}$ and let A_i be the local time at x_i subject to (6.3). Therefore if $A = \sum A_i$ the matrix U_A^λ defined in (5.1) is given by

$$U_A^\lambda(i, j) = U^\lambda(x_j - x_i), \tag{6.4}$$

and the matrix Q^λ appearing in Theorem 5.4 is $-(U_A^\lambda)^{-1}$. We will now apply Theorem 5.4 to obtain some explicit formulas.

6.5. THEOREM. *Let $a < b$ and let T be the first hitting time of the two point set $\{a, b\}$. In addition to (6.2) we assume either*

(i) $\int_{-\infty}^{\infty} [\psi_R(y)]^{-1}\, dy < \infty$

or

(ii) $U^\lambda(0) \to \infty$ *as* $\lambda \to 0$ *and*

$$G(x) = \int_{-\infty}^{\infty} (1 - e^{-iyx})\, [\psi(y)]^{-1}\, dy \tag{6.6}$$

exists absolutely for each x. If (i) holds, then

$$P^x[X(T) = a] = \frac{H(0)\, H(a - x) - H(b - x)\, H(a - b)}{H(0)^2 - H(a - b)\, H(b - a)}$$

where

$$H(x) = \int_{-\infty}^{\infty} e^{-iyx}[\psi(y)]^{-1}\, dy,$$

while if (ii) holds

$$P^x[X(T) = a] = \frac{G(a - b) - G(a - x) + G(b - x)}{G(a - b) + G(b - a)}$$

where G is given by (6.6).

PROOF. We confine our attention to case (ii) which is the most interesting, the argument in case (i) being similar but simpler. We apply the discussion preceding the statement of Theorem 6.5 to the three point set $\{a, b, x\}$. Inverting the matrix U_A^λ in (6.4) and applying Theorem 5.4 we see that $P^x[X(T) = a]$ is given by

$$\lim_{\lambda \to 0} \frac{U^\lambda(0)\, U^\lambda(a - x) - U^\lambda(b - x)\, U^\lambda(a - b)}{U^\lambda(0)^2 - U^\lambda(a - b)\, U^\lambda(b - a)}.$$

But under assumption (ii), $U^\lambda(0) - U^\lambda(z) \to (1/2\pi)\, G(z)$ as $\lambda \to 0$ for each z. Combining this with the fact that $U^\lambda(0) \to \infty$ as $\lambda \to 0$ yields $U^\lambda(z)\, [U^\lambda(0)]^{-1} \to 1$ as $\lambda \to 0$ for each z. Using these facts the limit above is easily evaluated and yields Theorem 6.5.

6.7. REMARK. Note that

$$G(b-x)-G(a-x)=\int_{-\infty}^{\infty} e^{iyx}\left[\frac{e^{-iya}-e^{-iyb}}{\psi(y)}\right]dy,$$

and since the function in square brackets is integrable the Riemann-Lebesgue lemma implies that

$$\lim_{x\to\pm\infty} P^x[X(T)=a]=\frac{G(a-b)}{G(a-b)+G(b-a)}.$$

6.8. COROLLARY. *Let X be a stable process of index $\alpha>1$, that is,*

$$\psi(y)=-|y|^\alpha\left[1+i\beta\operatorname{sgn}(y)\tan\frac{\pi\alpha}{2}\right]$$

where $1<\alpha<2$ and $-1\leqslant\beta\leqslant 1$, then using the same notations as in Theorem 6.5.

$$P^x[X(T)=a]=\frac{p(x)+\beta q(x)}{2(b-a)^{\alpha-1}}$$

where

$$p(x)=(b-a)^{\alpha-1}+|x-b|^{\alpha-1}-|x-a|^{\alpha-1}$$

and

$$q(x)=(b-a)^{\alpha-1}-\operatorname{sgn}(b-x)|x-b|^{\alpha-1}+\operatorname{sgn}(a-x)|x-a|^{\alpha-1}.$$

PROOF. It is immediate that X satisfies condition (ii) of Theorem 6.5. Thus we must compute G for the given ψ. A straightforward computation yields

$$G(x)=\frac{\pi[\beta\operatorname{sgn}(x)-1]}{(1+h^2)\,\Gamma(\alpha\ \cos(\pi\alpha/2)}|x|^{\alpha-1}$$

where $h=\beta\tan(\pi\alpha/2)$. Substituting this expression for G into the formula in Theorem 6.5 leads immediately to the conclusion of Corollary 6.8.

6.9. REMARK. A sketch of the graph of $x\to P^x[X(T)=a]$ is instructuve.

6.10. REMARK. Suppose X is symmetric, that is, $\beta=0$, and let $a=-1$, $b=1$; then the probability that X hits $\{-1\}$ before $\{1\}$ starting from x is given by

$$p_1(x)=\frac{2^{\alpha-1}+|x-1|^{\alpha-1}-|x+1|^{\alpha-1}}{2^\alpha}.$$

In [11] it was shown that the probability that X hits $(-\infty, -1]$ before $[1, \infty)$ is given by

$$p_2(x) = \begin{cases} 1, & x \leqslant -1 \\ 2^{1-\alpha}\Gamma(\alpha)\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^{-2} \int_x^1 (1-u^2)^{(\alpha/2)-1}\, du, & -1 < x < 1 \\ 0, & x \geqslant 1. \end{cases}$$

As one would expect these expressions agree if and only if $\alpha = 2$. Note that p_1 is correct when $\alpha = 2$, although the derivation of p_1 given here is valid only if $1 < \alpha < 2$. Of course, the whole problem is trivial when $\alpha = 2$.

One can now amuse oneself by computing various explicit formulas as corollaries of Theorem 5.4. We mention one more example. Let X be a symmetric stable process on the real line with index α, $1 < \alpha < 2$. Let T be the first hitting time of the three point set $\{-1, 0, 1\}$, then

$$P^x[X(T) = 0] = \frac{|1+x|^{\alpha-1} + |1-x|^{\alpha-1} - 2|x|^{\alpha-1} + 2 - 2^{\alpha-1}}{4 - 2^{\alpha-1}},$$

$$P^x[X(T) = 1] = \frac{\left\{\begin{array}{l} 2^{\alpha-1}[1 + |x|^{\alpha-1} - |1+x|^{\alpha-1}] \\ \qquad - 2[|1-x|^{\alpha-1} - |1+x|^{\alpha-1}] \end{array}\right\}}{2^{\alpha-1}[4 - 2^{\alpha-1}]},$$

and $P^x[X(T) = -1]$ is obtained from the last expression by symmetry. One can sketch these probabilities as functions of x; in particular for $|x|$ large the probabilities of hitting -1, 0, or 1 first are given approximately by $1, 2 - 2^{\alpha-1}$, and 1 divided by $4 - 2^{\alpha-1}$ respectively. It is perhaps interesting that starting from a large positive x the process is more likely to enter the set $\{-1, 0, 1\}$ at -1 than at 0.

Note Added in Proof. Of course, $(2\pi)^{-1} G$, where G is given by (6.6), is just the "potential kernel" for the recurrent process X. It is clear that the second conclusion of Theorem 6.5 is valid whenever $\lim_{\lambda \to 0} [U^\lambda(0) - U^\lambda(x)]$ exists if we call the limit $(2\pi)^{-1} G(x)$. It is also easy to see that if $g(x, y)$ is the Green's function (potential kernel) for X "killed" when it first hits zero, then

$$2\pi g(x, y) = G(-x) + G(y) - G(y - x).$$

These formulas should be compared with those obtained by Spitzer for recurrent random walks. See [12].

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